Debt policy rule, utility-generating government spending, and equilibrium (in)determinacy: A note*

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Abstract

This note uses an endogenous growth model to examine the effects of borrowing for public services that increase households’ utility (i.e., utility-generating government services). In an endogenous growth model of productive government services, Futagami, Iwaisako, and Ohdoi (2008) show that equilibrium indeterminacy occurs if the debt target is defined in terms of the debt-to-capital ratio. In contrast, Minea and Villieu (2013) show that indeterminacy disappears if the debt target is defined in terms of the debt-to-GDP ratio. In these two studies, the occurrence of indeterminacy depends only on the definition of the debt target. Here, we point out that the level of the debt target, the tax rate, and the household utility parameters are important determinants of indeterminacy when considering utility-generating government services.

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Keywords: Public Expenditure, Public Debt, Debt Policy Rule, Indeterminacy, Endogenous Growth

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*We are responsible for any remaining errors.
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1 Introduction

Assuming that government expenditure is financed by income tax and by issuing bonds in an endogenous growth model with productive government services, as proposed by Barro (1990), Futagami, Iwaisako, and Ohdoi (2008; hereafter, FIO) investigate the following government debt adjustment rule:

\[ \dot{b}_t = -\phi(b_t - \bar{b}), \]  

(1)

where \( b_t \) is defined as the public debt \((B_t)\)-to-private capital \((K_t)\) ratio, \( \phi > 0 \) is the adjustment coefficient, and \( \bar{b} > 0 \) is the target level of the government debt ratio.\(^1\) Under (1), the government gradually adjusts the public debt so that \( b_t \) becomes equal to the target level, \( \bar{b} \), in the long run.\(^2\) FIO show that (i) there are multiple balanced growth paths (BGPs) (i.e., two steady states) and (ii) indeterminacy of the transition path to the high-growth BGP occurs. Although FIO’s results are interesting, Minea and Villieu (2012; hereafter, MV) point out that if \( b_t \) is defined as the public debt \((B_t)\)-to-GDP \((Y_t)\) ratio, the results change dramatically. In fact, MV show that (i) there exists a unique steady state and (ii) indeterminacy is removed. In these two studies, whether the economy exhibits indeterminacy depends only how we define \( b_t \).

In this note, we introduce the government debt adjustment rule given by (1) into an endogenous growth model with the AK production function, where government spending is used for public services that increase households’ utility. Then, in contrast to the discussion among FIO and MV, we show that the level of the debt target, income tax rate, and household utility parameters are important determinants of indeterminacy when considering utility-generating government services. More precisely, we obtain the following three results: (i) there is a unique steady state; (ii) under some preference parameter conditions, there exists a threshold for the target level of public debt such that, when \( \bar{b} \) is

\(^1\)Maebayashi, Hori, and Futagami (2014) and Morimoto, Hori, Maebayashi, and Futagami (2013) also examine this debt policy rule in models in which the stock of public capital positively affects labor productivity.

\(^2\)The Maastricht Treaty requires that EU member states keep the government debt-to-GDP ratio below 60%. In the debt reduction benchmark (rule) introduced by the reform of the Stability and Growth Pact (SGP) in December 2011, the so-called Six-Pack, member states with a current debt-to-GDP ratio above 60% must reduce the distance to 60% by an average rate of one twentieth per year.
larger than the threshold, the steady state exhibits indeterminacy; and (iii) the threshold of the target level of public debt increases with the income tax rate.

2 Model

2.1 Firm

We consider a competitive economy. Time is continuous and denoted by \( t \geq 0 \). As in FIO and MV, we use an endogenous growth model. To focus on the interaction between households and the public sector, we model the production side of the economy as simply as possible. The representative firm produces a single final good using the following production function:

\[
Y_t = AK_t, \tag{2}
\]

where \( A \) is a positive constant, and \( Y_t \) and \( K_t \) represent output and capital input at time \( t \), respectively. Output grows at the same rate of \( K_t \). Through perfect competition and the profit maximization, the interest rate, \( r \), remains constant over time: \( r = A \).

2.2 Household

The economy is populated by identical households. The population size is constant over time and normalized to one. The representative household is endowed with an infinite lifetime and perfect foresight. We specify the utility of the representative household as

\[
U = \int_0^{\infty} \frac{u(C_t; G_t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \quad \sigma > 0, \tag{3}
\]

where

\[
u(C_t, G_t) = \begin{cases} [\theta_1 C_t^\eta + \theta_2 G_t^\eta]^{\frac{1}{\eta}}, & \eta < 1, \ \eta \neq 0, \\ C_t^{\theta_1} G_t^{\theta_2}, & \eta = 0. \end{cases} \tag{4}
\]

In (3), \( \rho > 0 \) and \( 1/\sigma \) denote the subjective discount rate and the intertemporal elasticity of substitution of \( u_t \), respectively. When \( \sigma = 1 \), the instantaneous utility function takes a
logarithmic form, \( \ln u_t \). In (4), \( C_t \) and \( G_t \) represent private consumption and the public services supplied by the government, respectively. In addition, \( \theta_1 \) and \( \theta_2 \) are positive constants that satisfy \( \theta_1 + \theta_2 = 1 \), and \( 1/(1-\eta) \) is the elasticity of substitution between \( C_t \) and \( G_t \) in \( u_t \). To ensure positive growth, we assume \( A > \rho \). As in Futagami et al. (2008) and Minea and Villieu (2013), the government levies taxes on the interest income. Then, the budget constraint of the household is given by \( \dot{W}_t = (1 - \tau)rW_t - C_t \), where \( W_t \) is the asset holding of the household and \( \tau \in [0, 1] \) is the interest income tax rate.

The utility maximization of the household yields \( \dot{\lambda}_t = \lambda_t[p - (1 - \tau)r] \) and

\[
[\theta_1 C_t^\eta + \theta_2 G_t^\eta]^{1-\frac{\sigma}{\eta}} \theta_1 C_t^{\eta - 1} = \lambda_t, \tag{5}
\]

where \( \lambda_t \) is the co-state variable associated with \( W_t \). From these two equations, we obtain

\[
\frac{\dot{g}_t}{g_t} = \frac{\sigma}{(1 + \theta x_t^\eta)^2} \frac{x_t}{x_t} + \frac{\dot{Y}_t}{Y_t} = \frac{1}{\sigma} [(1 - \tau)r - \rho], \tag{6}
\]

where \( g_t \equiv G_t/Y_t \), \( x_t \equiv G_t/C_t \equiv g_t/c_t \), \( c_t \equiv C_t/Y_t \), and \( \theta \equiv \theta_2/\theta_1 \). In addition, the transversality condition, \( \lim_{t \to \infty} \lambda_t W_t e^{-\rho t} = 0 \), must hold. The above equation shows that the growth rate of \( G_t \) affects the consumption and saving behavior of the household.

2.3 Government

As in Futagami et al. (2008) and Minea and Villieu (2013), the government finances its expenditure, \( G_t \), by levying interest income tax and issuing bonds. The government is allowed to run budget deficits. The budget constraint of the government is

\[
rB_t + G_t = \dot{B}_t + \tau rW_t, \tag{7}
\]

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\(^3\)Define \( v(C_t, G_t) \equiv u(C_t, G_t)^{(1-\sigma)/(1-\sigma)} \). We can verify the concavity of \( v \) with respect to \( C_t \) as follows: We have \( v_{cc}(C_t, G_t) = -\sigma u(C_t, G_t)^{-\sigma - 1} u_t u_t(C_t, G_t)^2 + u(C_t, G_t)^{-\sigma} u_{cc}(C_t, G_t) \). The first term is negative because \( u_{cc}(C_t, G_t) = \theta_1(\theta_1 C_t^\eta + \theta_2 G_t^\eta)(1/\eta)^{-1} C_t^{\eta - 1} > 0 \). The second term is also negative because \( u_{cc}(C_t, G_t) = -(1-\eta)C_t^{-1} u_t(C_t, G_t)^2 \theta_2 G_t^\eta / (\theta_1 C_t^\eta + \theta_2 G_t^\eta) < 0 \). Then, the intertemporal elasticity of substitution of \( C_t \) is \( v_{cc}(C_t, G_t) C_t/v_{cc}(C_t, G_t) C_t'(C_t, G_t) = \sigma u_{cc}(C_t, G_t) C_t/u(C_t, G_t) - u_{cc}(C_t, G_t) C_t/u(C_t, G_t) = [\sigma + (1-\eta)(\theta_2/\theta_1)(G_t/C_t)^\eta]/[1 + (\theta_2/\theta_1)(G_t/C_t)^\eta] \).
where \( B_t \) represents the outstanding debts of the government and \( \dot{B}_t \) denotes newly-issued government bonds. We assume that the government follows the debt adjustment rule given by (1). As in the MV model, we define \( b_t \equiv B_t / Y_t \). Then, the government attempts to adjust the debt-to-GDP ratio so that it becomes constant at \( \bar{b} \) in the long run.

### 3 Dynamic System

This section derives the dynamic system of the economy. The asset market equilibrium implies that \( W_t = K_t + B_t \). Using this equation together with (2), (7), and \( r = A \), we obtain \( \dot{B}_t = (1 - \tau)AB_t + G_t - \tau Y_t \). From the definition of \( b_t \), this equation is rearranged as

\[
\frac{\dot{b}_t}{b_t} = (1 - \tau)A + \frac{G_t}{B_t} - \frac{\tau}{b_t} - \frac{Y_t}{Y_t}.
\]

(8)

The goods market equilibrium condition is given by \( \dot{K}_t = Y_t - C_t - G_t \). From (1), (8), \( \dot{K}_t = Y_t - C_t - G_t \), and the definitions of \( g_t \) and \( x_t \), we obtain

\[
g_t = \frac{[(1 + Ab_t)\tau - \phi(b_t - \bar{b})]x_t}{(1 + Ab_t)x_t + Ab_t}.
\]

(9)

Differentiating both sides of the above equation with respect to \( t \) yields

\[
\frac{\dot{g}_t}{g_t} = \frac{Ab_t}{(1 + Ab_t)x_t + Ab_t} \frac{\dot{x}_t}{x_t} - \Psi(x_t, b_t),
\]

(10)

where \( \Psi(x_t, b_t) \equiv \left[ \frac{A\tau - \phi}{(1 + Ab_t)x_t + Ab_t} - \frac{A(1 + x_t)}{x_t} \right] \phi(b_t - \bar{b}) \). From (2) and \( \dot{K}_t = Y_t - C_t - G_t \), we have \( \frac{\dot{Y}_t}{Y_t} = \frac{\dot{K}_t}{K_t} = A \left[ 1 - \frac{1 + x_t}{x_t} g_t \right] \). Substituting this and (10) into (6) and using (9), we obtain

\[
\dot{x}_t = \left[ \frac{Ab_t}{(1 + Ab_t)x_t + Ab_t} - \frac{\sigma + (1 - \eta)\theta x_t^\eta}{\sigma(1 + \theta x_t^\eta)} \right]^{-1} \Gamma(x_t, b_t)x_t,
\]

(11)

\[\text{As a result of the AK production function, defining the ratio of the long-run debt target as } B_t / Y_t \text{ or } B_t / K_t \text{ does not affect the dynamics of this economy and does not change the results in this note.}\]
where
\[
\Gamma(x_t, b_t) \equiv \frac{1}{\sigma}[(1 - \tau)A - \rho] - A \left(1 - \frac{[(1 + Ab_t)\tau - \phi(b_t - \bar{b})](1 + x_t)}{(1 + Ab_t)x_t + Ab_t}\right) + \Psi(x_t, b_t).
\]

Given the initial value of \(b_t\), the dynamic system of the economy is composed of (1) and (11).

4 Steady State

This section derives the steady state where \(b_t\) and \(x_t\) are both constant over time and \(B_t, G_t, C_t,\) and \(K_t\) grow at the same constant rate. We omit the time index, \(t\), from those variables that are constant in the steady state and use an asterisk for a steady-state variable.

4.1 Existence

In the steady state, \(g_t\) becomes constant, from (9). Then, from (6), we immediately obtain the growth rate in the steady state, \(\gamma^* = \frac{1}{\sigma}[(1 - \tau)A - \rho]\). We set \(\dot{b}_t = 0\) in (1) to obtain \(b^* = \bar{b}\). Substituting \(\dot{x}_t = 0\) and \(b^* = \bar{b}\) into (11) yields
\[
x^* = \frac{(1 + A\bar{b})\tau - A\bar{b} \left(1 - \frac{\gamma^*}{A}\right)}{(1 + A\bar{b}) \left(1 - \tau - \frac{\gamma^*}{A}\right)} = \frac{A(\bar{b}_1 - \bar{b})}{1 + A\bar{b}},
\]
where \(\bar{b}_1 \equiv \tau / [A \left(1 - \tau - \frac{\gamma^*}{A}\right)]\). Inserting (12) and \(b^* = \bar{b}\) into (9) and \(c^* = g^*/x^*\) yields
\[
g^* = (1 + A\bar{b})\tau - A\bar{b} \left(1 - \frac{\gamma^*}{A}\right) = A \left(1 - \tau - \frac{\gamma^*}{A}\right)(\bar{b}_1 - \bar{b}),
\]
\[
c^* = (1 + A\bar{b}) \left(1 - \tau - \frac{\gamma^*}{A}\right).
\]
To ensure \(\bar{b}_1, c^*, g^*\) and \(x^* > 0\), we assume
\[
1 - \tau - \frac{\gamma^*}{A} > 0, \quad (13)
\]
\[
\bar{b} < \bar{b}_1. \quad (14)
\]
Based on the discussion so far, we obtain the following proposition.

**Proposition 1** Suppose that (13) and (14) are satisfied. There exists a unique steady-state equilibrium where \( b^* = \bar{b} \) holds, \( c^* \) and \( g^* \) are strictly positive, and \( B_t, G_t, C_t, \) and \( K_t \) grow at the rate of \( \gamma^* \).

### 4.2 Stability

To examine the local stability of the unique steady state, we linearly approximate the dynamic system around the steady state:

\[
\begin{pmatrix}
\dot{b}_t \\
\dot{x}_t
\end{pmatrix} =
\begin{pmatrix}
-\phi & 0 \\
\omega_1 x^* & \omega_2 x^*
\end{pmatrix}
\begin{pmatrix}
b_t - \bar{b} \\
x_t - x^*
\end{pmatrix},
\]

where

\[
\omega_1 \equiv f(\bar{b})^{-1} \Gamma_{b_t}(x_t, b_t)|_{(x_t, b_t) = (x^*, \bar{b})}, \quad \text{and} \quad \omega_2 \equiv -f(\bar{b})^{-1} A(1 + \bar{b}) \left( 1 - \frac{\tau^*}{\tau} \right)^2.
\]

Here, \( \Gamma_{b_t}(x_t, b_t) \) is the derivative of \( \Gamma(x_t, b_t) \) with respect to \( b_t \) and \( f(\bar{b}) \) is defined as

\[
f(\bar{b}) = \frac{A}{\tau} \left( 1 - \frac{\tau^*}{\tau} \right) \frac{\left[ (\bar{b} - \bar{b}_2) \theta(x^*)^\eta - (\bar{b}_1 - \bar{b}) \right]}{[1 + \theta(x^*)^\eta]},
\]

where \( \bar{b}_2 \equiv (1 - \eta) \tau/[\sigma A \left( 1 - \frac{\tau^*}{\tau} \right)] = (1 - \eta) \bar{b}_1/\sigma(>0) \). One of the two eigenvalues of the Jacobian matrix \(-\phi < 0\) is inevitably negative. The other, \( \omega_2 x^* \), has a sign opposite to that of \( f(\bar{b}) \). Note that \( b_t \) is a state variable, whereas \( x_t \) is a jump variable. Then, if \( f(\bar{b}) > 0 \) holds, the steady-state equilibrium exhibits local indeterminacy. In contrast, if \( f(\bar{b}) < 0 \), the steady-state equilibrium is saddle stable and locally determinate. Since (13) is assumed, the steady state is locally indeterminate (determinate) if and only if

\[
(\bar{b} - \bar{b}_2) \theta(x^*)^\eta > (<)(\bar{b}_1 - \bar{b}). \tag{15}
\]

When \( \sigma + \eta < 1 \), \( \bar{b}_1 < \bar{b}_2 \) holds by the definition of \( \bar{b}_2 \). Recall that \( \bar{b} < \bar{b}_1 \) is assumed to ensure \( g^* > 0 \). Then, when \( \sigma + \eta < 1 \), the left-hand side (LHS) of (15) is smaller than
the right-hand side (RHS). Therefore, the steady state is locally determinate.

When \( \sigma + \eta > 1 \), we have \( \bar{b}_1 > b_2 \). If \( \bar{b} \leq b_2 \), the LHS of (15) is smaller than the RHS. The steady state is again locally determinate. We next consider the case in which \( \bar{b}_2 < \bar{b} < \bar{b}_1 \) holds. Using (12), we can rewrite (15) as

\[
\theta \left[ \frac{1 + A\bar{b}}{A(b_1 - b)} \right]^{1-\eta} > \left( \frac{1 + A\bar{b}}{A(b - b_2)} \right). 
\]  

Let us denote the LHS and the RHS of (16) as \( \pi(\bar{b}) \) and \( \mu(\bar{b}) \), respectively. As \( \bar{b} \) increases from \( \bar{b}_2 \) to \( \bar{b}_1 \), \( \pi(\bar{b}) \) increases monotonically from \( \pi(\bar{b}_2)(< +\infty) \) to \( +\infty \) (see Figure 1).\(^5\) In contrast, as \( \bar{b} \) increases from \( \bar{b}_2 \) to \( \bar{b}_1 \), \( \mu(\bar{b}) \) decreases monotonically from \( +\infty \) to \( \mu(\bar{b}_1)(< +\infty) \).\(^6\) Therefore, there exists a unique value, \( \bar{b}_3 \in (\bar{b}_2, \bar{b}_1) \), that satisfies \( \mu(\bar{b}_3) = \pi(\bar{b}_3) \).

For \( \bar{b} \in (\bar{b}_2, \bar{b}_3) \), \( \pi(\bar{b}) < \mu(\bar{b}) \) holds. In this case, the steady state is determinate. When \( \bar{b} \in (\bar{b}_3, \bar{b}_1) \), \( \pi(\bar{b}) > \mu(\bar{b}) \) holds and the steady state is indeterminate. From the discussion so far, we obtain the following proposition.

\[ [\text{Figure 1}] \]

**Proposition 2** Suppose that (13) and (14) hold.

1. If \( \sigma + \eta < 1 \), the steady state is saddle stable and locally determinate for any \( \bar{b} \) satisfying (14).

2. If \( \sigma + \eta > 1 \), there exists a unique \( \bar{b}_3 \in (\bar{b}_2, \bar{b}_1) \).

   (a) If \( \bar{b} < \bar{b}_3 \), the steady state is saddle stable and locally determinate.

   (b) If \( \bar{b} > \bar{b}_3 \), the steady state is stable (sink) and exhibits local indeterminacy.

To understand the intuition behind proposition 2, we rewrite (6) as

\[
\frac{\sigma + (1 - \eta)\theta x_t^n \hat{C}_t}{1 + \theta x_t^n} \frac{\hat{C}_t}{C_t} = (1 - \tau)A - \rho + \frac{(1 - \sigma + \eta)\theta x_t^n \hat{G}_t}{1 + \theta x_t^n} \frac{\hat{G}_t}{G_t}. 
\]  

Here, we use the definition of \( x_t \equiv G_t/C_t \) and \( g_t \equiv G_t/Y_t \). Suppose that the economy is initially in the steady state and households expect an increase in the growth of government

\[ \pi'(\bar{b}) = (1 - \eta)A^{n - \theta}(1 + A\bar{b})(1 + A\bar{b})^{-\eta}(b_1 - b)^{\eta - 2} > 0 \] holds for \( b < b_1 \) because \( -\infty \leq \eta \leq 1 \).  
\[ \mu'(\bar{b}) = -(1 + b_2)/[A(\bar{b} - b_2)^2] < 0 \text{ for } b_2 < \bar{b}. \]
spending, $\dot{G}/G$. Assume $\sigma + \eta > 1$. Thus, (17) shows that an increase in the growth of government spending makes current consumption more attractive, in which case, households reduce savings for future consumption. Then, capital accumulation is depressed and the GDP growth slows down, which leads to the following three effects. First, note that the debt policy rule, (1), ensures $\dot{B}_t = \bar{b} \dot{Y}_t$, because the economy is initially in the steady state and hence $b_t = B_t/Y_t = \bar{b}$ holds. The equation $\dot{B}_t = \bar{b} \dot{Y}_t$ implies that when GDP growth decreases, the government must reduce its bond issuance today to follow the debt policy rule, (1). Then, from (7), the government also reduces its spending today. Second, a decrease in the issuance of government bonds reduces future outstanding government debt. Then, the government’s future interest payments also decrease, which loosens the government’s future budget constraint and has a positive effect on future government spending. Third, a reduction in household savings leads to decreases in future tax revenue for the government, which has negative effects on future government spending. The first and second effects are positive for the growth rate of government spending, while the third has a negative effect. When $\bar{b}$ is large, the first and second effects tend to be strong relative to the third effect. Then, the expectation of an increase in the growth of government spending becomes self-fulfilling and indeterminacy occurs when $\bar{b}$ is sufficiently large. If $\bar{b}$ is sufficiently small, the first two effects tend to be weak relative to the third. In this case, the expectation is not self-fulfilling and the steady state exhibits determinacy.

We next turn to the case of $\sigma + \eta < 1$. Here, the expectation of an increase in the growth of government spending makes future consumption more attractive. As a result, households increase saving, which accelerates capital accumulation. In this case, the above three effects work in the opposite directions to those described earlier. First, government spending today tends to increase because the government can increase its current bond issuance. Second, the government’s future interest payments increase, negatively affecting its future spending. Third, an increase in households’ savings has a positive effect on future tax revenue and future government spending. Since a rise in bond issuance today crowds out some capital accumulation, the third effect is weak relative to the other two when $\sigma + \eta < 1$. Here households’ expectations cannot be self-fulfilling.

In the rest of this subsection, we concentrate on the case when $\sigma + \eta > 1$ because $\bar{b}_3$ de-
pends on the income tax rate $\tau$, which affects the likelihood of equilibrium indeterminacy. Assuming $\sigma + \eta > 1$, we totally differentiate $\pi(\tilde{b}_3) = \mu(\tilde{b}_3)$.

$$
(p'(\tilde{b}_3) - \mu'(\tilde{b}_3))d\tilde{b}_3 = \frac{1 - \eta}{\sigma} \pi(\tilde{b}_3) \frac{d\tilde{b}_1}{d\tau} \left( \frac{1}{b_3 - b_2} + \frac{\sigma}{b_1 - b_3} \right) d\tau.
$$

In deriving the above equation, we use $\pi(\tilde{b}_3) = \mu(\tilde{b}_3)$ and $\tilde{b}_2 = (1 - \eta)\tilde{b}_1/\sigma$. As is clear from Figure 1, we have $p'(\tilde{b}_3) > \mu'(\tilde{b}_3)$ and $\pi(\tilde{b}_3) > 0$. The definition of $\tilde{b}_1$ implies $d\tilde{b}_1/d\tau > 0$. Since $\tilde{b}_3 \in (\tilde{b}_2, \tilde{b}_1)$, we have $d\tilde{b}_3/d\tau > 0$, giving us the next proposition.

**Proposition 3** Suppose that (13) and (14) are satisfied. When $\sigma + \eta > 1$ holds, the debt target ratio threshold, $\tilde{b}_3$, increases with the income tax rate $\tau$.

The intuition behind Proposition 3 is quite straightforward. A higher tax rate brings about an increase in revenue to the government. This causes the third effect, which has a more positive and stronger effect on future government. Then indeterminacy tends to disappear.

### 4.3 Numerical examples

In the current benchmark model, we assume that the only production factor is physical capital. To make the model more realistic and to provide interesting numerical examples, we modify the model so that the production of the final good requires labor input. As far as possible, we retain the notation used and the assumptions in the original benchmark model. Following Romer (1986), we specify the production function of firm $i$ as $Y_{i,t} = A(K_{i,t})^\alpha(L_{i,t})^{1 - \alpha}$, where $\alpha \in (0, 1)$ and $L_{i,t}$ and $K_{i,t}$ are the labor and capital inputs of firm $i$, respectively. The aggregate capital stock, $K_t = \sum_i K_{i,t}$, yields external effects. The budget constraints of the household and government are given by $W_t = (1 - \tau)r(W_t + w_t) - C_t$ and $rB_t + G_t = \dot{B}_t + \tau(rW_t + w_t)$, respectively, where $w_t$ is the wage rate. As shown in the Appendix, there exists a unique steady state under this modification. As in the benchmark model, there exist $\tilde{b}_1$ and $\tilde{b}_3$ such that equilibrium indeterminacy arises for $\tilde{b} \in (\tilde{b}_3, \tilde{b}_1)$, on the condition that the preference parameters satisfy $\sigma + \eta > 1$.

In our numerical examples, we set $\rho = 0.05$, $\theta_1 = \theta_2 = 0.5$, and $\alpha = 0.3$, and consider several values of $\sigma$, $\eta$, and $\tau$ (see Table 1). The value of $A$ is chosen so that the long-run
growth rate, \( \gamma^* = (1/\sigma)(1 - \tau)\alpha A - \rho \), becomes equal to 0.02, for the given values of \( \sigma \) and \( \tau \). Table 1 presents the values of \( \bar{b}_3 \). As shown in Proposition 3, \( \bar{b}_3 \) increases with \( \tau \). In addition, when \( \sigma \) is large, \( \bar{b}_3 \) is small and indeterminacy tends to occur.

(1) \( \sigma = 2 \)  
<table>
<thead>
<tr>
<th>( \eta = -1 )</th>
<th>( \tau = 0.1 ) &amp; ( \tau = 0.2 )</th>
<th>( \tau = 0.3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7653</td>
<td>1.7104</td>
<td>2.788</td>
</tr>
</tbody>
</table>

| \( \eta = -0.5 \) | 0.66379 | 1.4412 | 2.3383 |
|---|---|---|
| \( \eta = 0 \) | 1.0061 | 1.8374 | 2.6714 |

| \( \eta = 0.5 \) | 0.22845 | 0.54526 | 0.9684 |

Table 1: Values of \( \bar{b}_3 \)

(2) \( \sigma = 4 \)  
| \( \eta = -1 \) | \( \tau = 0.1 \) & \( \tau = 0.2 \) & \( \tau = 0.3 \) |
|---|---|---|
| 0.21302 | 0.52155 | 0.92453 |

| \( \eta = -0.5 \) | 0.29839 | 0.64313 | 1.0475 |
|---|---|---|
| \( \eta = 0 \) | 0.60043 | 1.0614 | 1.5045 |

| \( \eta = 0.5 \) | 0.21302 | 0.52155 | 0.92453 |

5 Conclusion

In endogenous growth models with productive government services, FIO and MV examine the effects of the debt-adjustment rule specified by (1). According to these studies, whether indeterminacy occurs depends only on the definition of the public debt target. In contrast, we show that when considering utility-generating government services, the level of the debt target, the income tax rate, and the preference parameters play important roles in indeterminacy. More precisely, we find that under some conditions for the preference parameters, there exists a threshold such that if the target debt level is larger than the threshold, the steady state exhibits indeterminacy. In addition, the threshold increases with the income tax rate.

Appendix

The profit maximization for firms yields wage rate \( w_t = (1 - \alpha)AK_t \) and interest rate \( r = \alpha A \), respectively. The aggregate production function is written as \( Y_t = AK_t \) because \( \sum_i L_{i,t} = 1 \). The first-order condition for the optimization of households remains unchanged. Using the government budget constraint, the resource constraint, \( Y_t = AK_t \),
\[ w_t = (1 - \alpha)AK_t, \quad r = \alpha A, \quad \text{and} \quad (6), \quad \text{we can modify} \quad (11) \quad \text{as follows} \]

\[ \dot{x}_t = \left[ \frac{Ab_t}{(1 + Ab_t)x_t + Ab_t} - \frac{\sigma + (1 - \eta)\theta x_t^\eta}{\sigma(1 + \theta x_t^\eta)} \right]^{-1} \tilde{\Gamma}(x_t, b_t)x_t, \quad (18) \]

where

\[ \tilde{\Gamma}(x_t, b_t) \equiv \frac{1}{\sigma}[(1 - \tau)\alpha A - \rho] - A \left( 1 - \frac{\tau + A[1 - (1 - \tau)\alpha]b_t - \phi(b_t - \tilde{b})](1 + x_t)}{(1 + Ab_t)x_t + Ab_t} \right) + \Psi(x_t, b_t). \]

In the steady state, the long-run growth rate is given by \[ \gamma^* = \frac{1}{\sigma}[(1 - \tau)\alpha A - \rho], \]
and we have \[ x^* = A(\bar{b} - \tilde{b}_1)/(\Omega + A\bar{b}) \]
and \[ \tilde{b}_1 = \tau/[A\{(1 - \tau)\alpha - \gamma^*/A\}], \]
where \( \Omega = \{1 - \tau - \gamma^*/A\}/\{(1 - \tau)\alpha - \gamma^*/A\}. \)

The inequalities \( \bar{b} < \tilde{b}_1 \) and \( 1 - \tau - \gamma^*/A > 0 \) ensure that \( x^* > 0 \) and \( \tilde{b}_1 > 0. \) The steady state is locally indeterminate (determinate) if and only if

\[ \left\{ \left[ \frac{\sigma + \eta - 1}{1 + Ab} + \frac{1 - \eta}{\Omega + Ab} \right] \bar{b} - \frac{(1 - \eta)\tilde{b}_1}{\Omega + Ab} \right\} \theta(x^*)^{\eta} > (<) \frac{\sigma(\tilde{b}_1 - \bar{b})}{\Omega + Ab}. \quad (19) \]

When \( \sigma + \eta < 1, \) we have \[ \left[ \frac{\sigma + \eta - 1}{1 + Ab} + \frac{1 - \eta}{\Omega + Ab} \right] \bar{b} < \frac{1 - \eta}{\Omega + Ab} \bar{b} < \frac{1 - \eta}{\Omega + Ab} \tilde{b}_1. \]
Since the LHS is negative, indeterminacy never arises. We next consider the case when \( \sigma + \eta > 1. \) As \( \bar{b} \) increases from zero to \( \tilde{b}_1, \) the first term in curly brackets on the LHS increases monotonically from zero to a constant larger than \( \frac{(1 - \eta)\tilde{b}_1}{\Omega + Ab}, \) while the second term decreases monotonically. Then, there exists a unique \( \tilde{b}_2(< \tilde{b}_1) \) such that the LHS of (19) is strictly positive for \( \bar{b} \in (\tilde{b}_2, \tilde{b}_1). \)

Assuming \( \sigma + \eta > 1 \) and \( \bar{b} \in (\tilde{b}_2, \tilde{b}_1), \) we rearrange (19) as

\[ \theta \left[ \frac{\Omega + A\bar{b}}{A(\tilde{b}_1 - \bar{b})} \right]^{1 - \eta} > (<) \frac{\sigma}{A} \left\{ \left[ \frac{\sigma + \eta - 1}{1 + Ab} + \frac{1 - \eta}{\Omega + Ab} \right] \bar{b} - \frac{(1 - \eta)\tilde{b}_1}{\Omega + Ab} \right\}^{-1}. \quad (20) \]

As \( \bar{b} \) increases from \( \tilde{b}_2 \) to \( \tilde{b}_1, \) the LHS of (20) increases monotonically from a constant to \( +\infty. \) In contrast, as \( \bar{b} \) increases from \( \tilde{b}_2 \) to \( \tilde{b}_1, \) the RHS decreases monotonically from \( +\infty \) to a constant. Therefore, there exists a unique value \( \tilde{b}_3 \in (\tilde{b}_2, \tilde{b}_1) \) such that, when \( \bar{b} \in (\tilde{b}_3, \tilde{b}_1), \) the steady state is indeterminate.
References


Figure 1: Indeterminacy or Determinacy when $\sigma + \eta > 1$